1. The Number e

1.1. **Continuous Compound Interest.** We wish to define continuously compounded interest as the limit of periodically compounded interest as the k goes to infinity. Thus we fix A_0, r, and t, and attempt to understand the expression

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}.$$

To do this, we define a new variable n by $n=\frac{k}{r}$, so that $k=nr$ and $\frac{r}{k}=\frac{1}{n}$. Since r is fixed, n goes to infinity as k goes to infinity. We compute

$$\begin{aligned}\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} &= \lim_{n\to\infty} A_0\left(1+\frac{1}{n}\right)^{nrt}\\ &= \lim_{n\to\infty} A_0\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}\\ &= A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.\end{aligned}$$

This computation tells us that continuously compounded interest may be computed using an exponential function whose base is the limit of the sequence $(1+\frac{1}{n})^n$; it can be show that this is an increasing sequence which is bounded above by 3, so it converges. The number it converges to turns out to be so important in mathematics that we give it a special name.

Define

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Then, the equation which computes the amount A_t for continuously compounded interest is

$$A_t = A_0 e^{rt}.$$

We estimate e by computing a few values:

n	$(1+\frac{1}{n})^n$	estimate
1	$(2)^1$	2.000000
2	$(1.5)^2$	2.250000
4	$(1.25)^4$	2.441406
10	$(1.1)^{10}$	2.593742
100	$(1.01)^{100}$	2.704813
1000	$(1.001)^{1000}$	2.716923
10000	$(1.0001)^{10000}$	2.718145
100000	$(1.00001)^{100000}$	2.718268
∞	e	2.718281